Fusion and analytical Bethe ansatz for the $A_{N-1}^{(1)}$ open spin chain

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Abstract. We generalize the fusion procedure for the $A^{(1)}_{\mathcal{N}-1}$ open spin chain ($\mathcal{N} > 2$) and we show that the transfer matrix satisfies a crossing property. We use these results to solve the $A^{(1)}_{\mathcal{N}-1}$ open spin chain with $U_q(SU(\mathcal{N}))$ symmetry by means of the analytical Bethe ansatz method. Our results coincide with the known ones obtained via the nested Bethe ansatz.

1. Introduction

We consider the $A^{(1)}_{\mathcal{N}-1}$ [1, 2] open spin chain ($\mathcal{N} > 2$) with N sites. This is an integrable system which has been solved by means of the nested Bethe ansatz [3, 4]. Also, this model, in the critical regime, can be thought as the lattice analogue of a certain two-dimensional field theory, i.e. the $A^{(1)}_{\mathcal{N}-1}$ affine Toda field theory [5]. We focus here on the special case where the chain has $U_q(SU(\mathcal{N}))$ symmetry [6,7], and we solve it using a simpler method compared with nesting, namely, the analytical Bethe ansatz [8, 9].

The analytical Bethe ansatz has been used for models with crossing symmetry, e.g. $A^{(1)}_1$, $A^{(1)}_{2n}$ (see [9, 10]). This is the first time that this method has been used for a model without crossing symmetry, i.e. the $A^{(1)}_{\mathcal{N}-1}$ open spin chain. Although this spin chain does not have such symmetry, the corresponding R-matrix satisfies a crossing property [11–14], and consequently one can show that the transfer matrix satisfies an analogous property. One can also generalize the results of [15, 16] and derive a fusion procedure for the corresponding open chain transfer matrix. The crossing property, the fusion of the transfer matrix and the quantum group symmetry play an essential role in the derivation of the analytical Bethe ansatz.

The outline of the paper is as follows. In section 2 we describe the model and we derive the crossing property for the R-matrix and the transfer matrix. In the next section we deduce the fusion procedure for the open chain transfer matrix. In section 4 we derive the asymptotic behaviour of the transfer matrix and together with the results of the previous sections, periodicity and analyticity we find the spectrum of the transfer matrix and the Bethe ansatz equations. We illustrate the method using the $A^{(1)}_2$ chain, but the results can nevertheless be generalized for any $\mathcal{N}$. Finally, in the last section we give a brief discussion of the results of this work.

2. The model

There are two basic building blocks for constructing open spin chains:

(1) The R-matrix, which is a solution of the Yang–Baxter equation

$$R_{12}(\lambda_1 - \lambda_2) \, R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) \, R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2) \qquad (2.1)$$

(see, e.g., [17]). We assume that the R-matrix has the unitarity property

$$R_{12}(\lambda) R_{21}(-\lambda) = \zeta(\lambda) \qquad (2.2)$$

where $R_{21}(\lambda) = \mathcal{P}_{12} R_{12}(\lambda) \mathcal{P}_{12} = R_{12}(\lambda)^{t_1 t_2}$, t denotes transpose and $\mathcal{P}$ is the permutation matrix, and also the property [18]

$$R_{12}(\lambda)^{t_1} M_1 R_{12}(-\lambda - 2\rho)^{t_2} M_1^{-1} = \zeta'(\lambda) \qquad (2.3)$$

with $M^t = M$,

$$[M_1 M_2, R_{12}(\lambda)] = 0 \qquad (2.4)$$

and

$$\zeta(\lambda) = \sinh \mu(\lambda + \mathrm{i}) \sinh \mu(-\lambda + \mathrm{i}) \qquad \zeta'(\lambda) = \sinh \mu(\lambda + \rho) \sinh \mu(-\lambda + \rho). \quad (2.5)$$

For the purposes of this work we are going to need also the R-matrix that involves different representations of $U_q(SU(\mathcal{N}))$ [19, 20], in particular, $\mathcal{N}$ and $\bar{\mathcal{N}}$. This matrix is given by crossing

$$R_{\bar{1}2}(\lambda) = V_1 \, R_{12}(-\lambda - \rho)^{t_2} \, V_1 = V_2^{t_2} \, R_{12}(-\lambda - \rho)^{t_1} \, V_2^{t_2} \qquad (2.6)$$

where $V^2 = 1$ and $M = V^t V$, for the $A_1^{(1)}$ case $R_{\bar{1}2}(\lambda) = R_{12}(\lambda)$. $R_{\bar{1}2}(\lambda)$ also satisfies the unitarity property,

$$R_{\bar{1}2}(\lambda) R_{2\bar{1}}(-\lambda) = \zeta'(\lambda) \qquad (2.7)$$

this equation is equivalent to (2.3), with $R_{2\bar{1}}(\lambda) = R_{\bar{1}2}(\lambda)^{t_1 t_2}$. Moreover

$$R_{\bar{1}2}(\lambda)^{t_1} M_1 R_{\bar{1}2}(-\lambda - 2\rho)^{t_2} M_1^{-1} = \zeta(\lambda) \qquad (2.8)$$

which is equivalent to (2.2). $R_{\bar{1}2}(\lambda)$ is also a solution of the Yang–Baxter equation

$$R_{\bar{1}2}(\lambda_1 - \lambda_2) R_{\bar{1}3}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{\bar{1}3}(\lambda_1) R_{\bar{1}2}(\lambda_1 - \lambda_2). \qquad (2.9)$$

(2) The matrices K^- and K^+, which are solutions of the boundary Yang–Baxter equation [21]:

$$R_{12}(\lambda_1 - \lambda_2) K_1^-(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2^-(\lambda_2)$$
$$= K_2^-(\lambda_2) \, R_{12}(\lambda_1 + \lambda_2) K_1^-(\lambda_1) \, R_{21}(\lambda_1 - \lambda_2) \qquad (2.10)$$

and

$$R_{12}(-\lambda_1 + \lambda_2) K_1^+(\lambda_1)^{t_1} M_1^{-1} R_{21}(-\lambda_1 - \lambda_2 - 2\rho) M_1 K_2^+(\lambda_2)^{t_2}$$
$$= K_2^+(\lambda_2)^{t_2} \, M_1 R_{12}(-\lambda_1 - \lambda_2 - 2\rho) \, M_1^{-1} K_1^+(\lambda_1)^{t_1} R_{21}(-\lambda_1 + \lambda_2). \qquad (2.11)$$

There exists an automorphism between K^- and K^+, i.e.

$$K^+(\lambda) = M K^-(-\lambda - \rho)^t. \qquad (2.12)$$

For the following we are going to need a reflection equation that involves $R_{\bar{1}2}$ as well, in particular,

$$R_{\bar{1}2}(\lambda_1 - \lambda_2) K_{\bar{1}}^-(\lambda_1) R_{2\bar{1}}(\lambda_1 + \lambda_2) \, K_2^-(\lambda_2)$$
$$= K_2^-(\lambda_2) R_{\bar{1}2}(\lambda_1 + \lambda_2) K_{\bar{1}}^-(\lambda_1) R_{2\bar{1}}(\lambda_1 - \lambda_2) \qquad (2.13)$$

and

$$R_{\bar{1}2}(-\lambda_1 + \lambda_2) K_{\bar{1}}^+(\lambda_1)^{t_1} M_1^{-1} R_{2\bar{1}}(-\lambda_1 - \lambda_2 - 2\rho) M_1 K_2(\lambda_2)^{t_2}$$
$$= K_2^+(\lambda_2)^{t_2} M_1 R_{\bar{1}2}(-\lambda_1 - \lambda_2 - 2\rho) M_1^{-1} K_{\bar{1}}^+(\lambda_1)^{t_1} R_{2\bar{1}}(-\lambda_1 + \lambda_2). \qquad (2.14)$$

In the scattering language if we think that the K_i matrix describes the scattering of a soliton with the boundary, then $K_{\bar{i}}$ describes the scattering of an anti-soliton with the boundary. Subsequently $R_{\bar{1}2}$ describes the scattering of a soliton with an anti-soliton.

The corresponding transfer matrix $t(\lambda)$ for an open chain of N spins is given by [22, 23]

$$t(\lambda) = \text{tr}_0 \, K_0^+(\lambda) T_0(\lambda) K_0^-(\lambda) \hat{T}_0(\lambda) \tag{2.15}$$

where tr_0 denotes the trace over the 'auxiliary space' 0, $T_0(\lambda)$ is the monodromy matrix

$$T_0(\lambda) = R_{0N}(\lambda) \dots R_{01}(\lambda) \tag{2.16}$$

and $\hat{T}_0(\lambda)$ is given by

$$\hat{T}_0(\lambda) = R_{10}(\lambda) \dots R_{N0}(\lambda). \tag{2.17}$$

(As is customary, we usually suppress the 'quantum-space' subscripts $1, \dots, N$.) Indeed, it can be shown that this transfer matrix has the commutativity property

$$[t(\lambda), t(\lambda')] = 0. \tag{2.18}$$

In this paper, we consider the case of the $A_{\mathcal{N}-1}^{(1)}$ R-matrix [24]

$$
\begin{aligned}
R_{12}(\lambda)_{jj,jj} &= \sinh \mu (\lambda + \text{i}) \\
R_{12}(\lambda)_{jk,jk} &= \sinh(\mu\lambda) \qquad j \neq k \\
R_{12}(\lambda)_{jk,kj} &= \sinh(\text{i}\mu)\text{e}^{\mu\lambda\text{sign}(j-k)} \qquad j \neq k \\
1 &\leqslant j \qquad k \leqslant \mathcal{N}
\end{aligned}
\tag{2.19}
$$

which depends on the anisotropy parameter $\mu \geqslant 0$, and which becomes $SU(\mathcal{N})$ invariant for $\mu \to 0$. This R-matrix has the properties (2.2) and (2.3), with [25]

$$M_{jk} = \delta_{jk}\text{e}^{-\text{i}\mu(\mathcal{N}-2j+1)} \qquad \rho = \text{i}\mathcal{N}/2. \tag{2.20}$$

The corresponding open spin chain Hamiltonian $\mathcal{H}$ for $K^-(\lambda) = 1$ and $K^+(\lambda) = M$ is

$$\mathcal{H} = \sum_{n=1}^{N-1} \mathcal{H}_{nn+1} + \frac{\text{tr}_0 \, M_0 \mathcal{H}_{N0}}{\text{tr} \, M} \tag{2.21}$$

where the two-site Hamiltonian $\mathcal{H}_{jk}$ is given by

$$\mathcal{H}_{jk} = \frac{\text{i}}{2} \mathcal{P}_{jk} \frac{\text{d}}{\text{d}\lambda} R_{jk}(\lambda) \bigg|_{\lambda=0}. \tag{2.22}$$

One can verify that the Hamiltonian is Hermitian.

We consider the case where $K^-(\lambda) = 1$ and $K^+(\lambda) = M$, and so the transfer matrix is $U_q(SU(\mathcal{N}))$ invariant [6, 7]. Following [9] we show that the transfer matrix satisfies a crossing property. To prove the crossing property we need (2.1), (2.3) and the following identity:

$$\mathcal{P}_{12}^{t_2} M_2 \, R_{12}(\lambda)^{t_1} = R_{12}(\lambda)^{t_1} M_1^{-1} \mathcal{P}_{12}^{t_2}. \tag{2.23}$$

It is important to mention that in order to show (2.23) we considered the 'unusual' reflection equation (2.13) for $\lambda_1 - \lambda_2 = -\rho$. Then the crossing property for the transfer matrix is given by

$$t(\lambda) = \bar{t}(-\lambda - \rho) \tag{2.24}$$

where

$$\bar{t}(\lambda) = \text{tr}_0 \, M_0 T_{\bar{0}}(\lambda) \hat{T}_{\bar{0}}(\lambda) \tag{2.25}$$

and

$$
\begin{aligned}
T_{\bar{0}}(\lambda) &= R_{\bar{0}N}(\lambda) \dots R_{\bar{0}1}(\lambda) \\
\hat{T}_{\bar{0}}(\lambda) &= R_{1\bar{0}}(\lambda) \dots R_{N\bar{0}}(\lambda).
\end{aligned}
\tag{2.26}
$$

The proof of equation (2.24) follows exactly the proof in [9]. The only difference is that in this case $\bar{t}(\lambda)$ is involved as well because of (2.6). The transfer matrix $\bar{t}(\lambda)$ satisfies the commutativity property

$$[\bar{t}(\lambda), \bar{t}(\lambda')] = 0. \tag{2.27}$$

Relation (2.24) is one of the basic results of this paper and it plays an essential role in the derivation of the transfer matrix eigenvalues. The 'new' transfer matrix (see also, e.g., [19]) leads apparently to a non-local Hamiltonian; however, this is not a problem since $\bar{t}(\lambda)$ has an auxiliary character in our calculations as we shall see later.

3. Fusion

The fusion procedure for spin chains with crossing symmetry is known [16]. We generalize this procedure for the case where the R-matrix does not have crossing symmetry. From now on the indices 1 and 2 refer to the auxiliary space. We consider the equation (2.6) for $\lambda = -\rho$. Then $R_{\bar{1}2}(\lambda)$ degenerates to a projector onto a one-dimensional subspace

$$P_{\bar{1}2}^- = \frac{1}{\mathcal{N}} V_1 \mathcal{P}_{12}^{t_2} V_1. \tag{3.1}$$

Also

$$P_{\bar{1}2}^+ = 1 - P_{\bar{1}2}^- \tag{3.2}$$

is a projector. We consider the Yang–Baxter equation (2.9) for $\lambda = -\rho$, then the fused R-matrix is given by

$$\begin{aligned} R_{\langle\bar{1}2\rangle 3}(\lambda) &= P_{\bar{1}2}^+ R_{\bar{1}3}(\lambda) R_{23}(\lambda + \rho) P_{\bar{1}2}^+ \\ R_{\langle 2\bar{1}\rangle 3}(\lambda) &= P_{2\bar{1}}^+ R_{23}(\lambda) R_{\bar{1}3}(\lambda + \rho) P_{2\bar{1}}^+. \end{aligned} \tag{3.3}$$

Also, one finds

$$\begin{aligned} R_{3\langle\bar{1}2\rangle}(\lambda) &= P_{\bar{1}2}^+ R_{32}(\lambda - \rho)\, R_{3\bar{1}}(\lambda)\, P_{\bar{1}2}^+ \\ R_{3\langle 2\bar{1}\rangle}(\lambda) &= P_{2\bar{1}}^+ R_{3\bar{1}}(\lambda - \rho) R_{32}(\lambda) P_{2\bar{1}}^+. \end{aligned} \tag{3.4}$$

Similarly, one can fuse the spaces 1 and $\bar{2}$. The fused R-matrices obey the general Yang–Baxter equation:

$$R_{j_1 j_2}(\lambda_1) R_{j_1 j_3}(\lambda_1 + \lambda_2) R_{j_2 j_3}(\lambda_2) = R_{j_2 j_3}(\lambda_2)\, R_{j_1 j_3}(\lambda_1 + \lambda_2) R_{j_1 j_2}(\lambda_1). \tag{3.5}$$

Consider the reflection equation (2.13) for $\lambda_1 - \lambda_2 = -\rho$, then the fused K-matrices are given by

$$\begin{aligned} K_{\langle\bar{1}2\rangle}^-(\lambda) &= P_{\bar{1}2}^+ K_{\bar{1}}^-(\lambda) R_{2\bar{1}}(2\lambda + \rho) K_2^-(\lambda + \rho) P_{2\bar{1}}^+ \\ K_{\langle\bar{1}2\rangle}^+(\lambda)^{t_{12}} &= P_{2\bar{1}}^+ K_{\bar{1}}^+(\lambda)^{t_1} M_2 R_{2\bar{1}}(-2\lambda - 3\rho) M_2^{-1} K_2^+(\lambda + \rho)^{t_2} P_{\bar{1}2}^+. \end{aligned} \tag{3.6}$$

The above K-matrices obey the reflection equations:

$$\begin{aligned} R_{3\langle\bar{1}2\rangle}(\lambda_1 - \lambda_2) K_3^-(\lambda_1) R_{\langle\bar{1}2\rangle 3}(\lambda_1 + \lambda_2)\, K_{\langle\bar{1}2\rangle}^-(\lambda_2) \\ = K_{\langle\bar{1}2\rangle}^-(\lambda_2) R_{3\langle 2\bar{1}\rangle}(\lambda_1 + \lambda_2 + \rho) K_3^-(\lambda_1) R_{\langle 2\bar{1}\rangle 3}(\lambda_1 - \lambda_2 - \rho) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} R_{\langle\bar{1}2\rangle 3}(-\lambda_1 + \lambda_2)^{t_{123}} K_3^+(\lambda_1)^{t_3} M_3^{-1} R_{3\langle\bar{1}2\rangle}(-\lambda_1 - \lambda_2 - 2\rho)^{t_{123}} M_3 K_{\langle\bar{1}2\rangle}^+(\lambda_2)^{t_{12}} \\ = K_{\langle\bar{1}2\rangle}^+(\lambda_2)^{t_{12}} M_3 R_{\langle 2\bar{1}\rangle 3}(-\lambda_1 - \lambda_2 - 3\rho)^{t_{123}} \\ \times M_3^{-1} K_3^+(\lambda_1)^{t_3} R_{3\langle 2\bar{1}\rangle}(-\lambda_1 + \lambda_2 + \rho)^{t_{123}}. \end{aligned} \tag{3.8}$$

Analogously we obtain the $K_{\langle 1\bar{2}\rangle}(\lambda)$-matrices.

Having fused the R- and K-matrices we can show that the corresponding fused transfer matrix is (for a detailed computation see, e.g., [16])

$$\hat{t}(\lambda) = \zeta(2\lambda + 2\rho)\bar{t}(\lambda)t(\lambda + \rho) - \Delta[K^+(\lambda)]\,\delta[T(\lambda)]\Delta[K^-(\lambda)]\,\delta[\hat{T}(\lambda)]. \tag{3.9}$$

Notice that $\bar{t}(\lambda)$ appears in the last equation as well as in (2.24). In this case we fuse the spaces $\bar{1}2$, and the quantum determinants are [16, 22]

$$\begin{aligned}
\delta[T(\lambda)] &= \mathrm{tr}_{12}[P_{\bar{1}2}^- T_{\bar{1}}(\lambda)\hat{T}_2(\lambda + \rho)] \\
\delta[\hat{T}(\lambda)] &= \mathrm{tr}_{12}[P_{2\bar{1}}^- T_{\bar{1}}(\lambda)\hat{T}_2(\lambda + \rho)] \\
\Delta[K^-(\lambda)] &= \mathrm{tr}_{12}[P_{\bar{1}2}^- K_1^-(\lambda)R_{2\bar{1}}(2\lambda + \rho)K_2^-(\lambda + \rho)V_1 V_2] \\
\Delta[K^+(\lambda)] &= \mathrm{tr}_{12}[P_{\bar{1}2}^- V_1 V_2 K_2^+(\lambda + \rho)M_2^{-1} R_{\bar{1}2}(-2\lambda - 3\rho)M_2\, K_1^+(\lambda)].
\end{aligned} \tag{3.10}$$

We can derive relations similar to (3.9) and (3.10) for the $1\bar{2}$ fusion. Using unitarity and the crossing property we prove the identities

$$\begin{aligned}
P_{\bar{1}2}^- R_{\bar{1}m}(\lambda)R_{2m}(\lambda + \rho)\,P_{\bar{1}2}^- &= \zeta(\lambda + \rho)P_{\bar{1}2}^- \\
P_{1\bar{2}}^- R_{1m}(\lambda)R_{\bar{2}m}(\lambda + \rho)P_{1\bar{2}}^- &= \zeta'(\lambda + \rho)P_{1\bar{2}}^- \\
m &= 1, 2, \ldots, N
\end{aligned} \tag{3.11}$$

and by computing the quantum determinants explicitly we find

$$\delta[T(\lambda)] = \delta[\hat{T}(\lambda)] = \zeta(\lambda + \rho)^N \qquad \text{or} \qquad \zeta'(\lambda + \rho)^N \tag{3.12}$$

depending on the spaces we fuse, i.e. $\bar{1}2$ or $1\bar{2}$ respectively. For the special case $K^-(\lambda) = 1$ and $K^+(\lambda) = M$, these are solutions of the reflection equations (2.13) and (2.14) correspondingly; one can show from (3.10) that

$$\Delta[K^-(\lambda)] = g(2\lambda + \rho) \qquad \Delta[K^+(\lambda)] = g(-2\lambda - 3\rho) \tag{3.13}$$

where

$$g(\lambda) = \sinh \mu(-\lambda + \rho). \tag{3.14}$$

4. Analytical Bethe ansatz

We focus here on the simplest case, namely, the $A_2^{(1)}$ open chain. The asymptotic behaviour of the R-matrix for $\lambda \to -\infty$ follows from (2.19):

$$\begin{aligned}
R_{0k} &\sim -\tfrac{1}{2}e^{-\mu\lambda}\begin{pmatrix} e^{-i\mu S_{1,k}} & pJ_{1,k}^- & pJ_{3,k}^- \\ 0 & e^{-i\mu S_{2,k}} & pJ_{2,k}^- \\ 0 & 0 & e^{-i\mu S_{3,k}} \end{pmatrix} \\
R_{k0} &\sim -\tfrac{1}{2}e^{-\mu\lambda}\begin{pmatrix} e^{-i\mu S_{1,k}} & 0 & 0 \\ pJ_{1,k}^+ & e^{-i\mu S_{2,k}} & 0 \\ pJ_{3,k}^+ & pJ_{2,k}^+ & e^{-i\mu S_{3,k}} \end{pmatrix}
\end{aligned} \tag{4.1}$$

where $p = 2\sinh(-i\mu)$, and the matrix elements are

$$\begin{aligned}
S_i &= e_{i,i} & i &= 1, 2, 3 \\
J_i^+ &= e_{i,i+1} & J_i^- &= e_{i+1,i} & i &= 1, 2 \\
J_3^+ &= e_{1,3} & J_3^- &= e_{3,1}
\end{aligned} \tag{4.2}$$

with

$$(e_{i,j})_{kl} = \delta_{ik}\delta_{jl}. \tag{4.3}$$

The leading asymptotic behaviour of the monodromy matrix is given by

$$T_0(\lambda) \sim (-\tfrac{1}{2})^N e^{-\mu\lambda N} \begin{pmatrix} e^{-i\mu\mathcal{S}_1} & p\mathcal{J}_1^- & p\mathcal{J}_3^- \\ 0 & e^{-i\mu\mathcal{S}_2} & p\mathcal{J}_2^- \\ 0 & 0 & e^{-i\mu\mathcal{S}_3} \end{pmatrix}$$

$$\hat{T}_0(\lambda) \sim (-\tfrac{1}{2})^N e^{-\mu\lambda N} \begin{pmatrix} e^{-i\mu\mathcal{S}_1} & 0 & 0 \\ p\mathcal{J}_1^+ & e^{-i\mu\mathcal{S}_2} & 0 \\ p\mathcal{J}_3^+ & p\mathcal{J}_2^+ & e^{-i\mu\mathcal{S}_3} \end{pmatrix} \tag{4.4}$$

where

$$\mathcal{S}_i = \sum_{k=1}^N S_{i,k} \qquad i = 1, 2, 3$$

$$\mathcal{J}_1^\pm = \sum_{k=1}^N e^{-i\mu S_{1,N}} \ldots e^{-i\mu S_{1,k+1}} J_{1,k}^\pm e^{-i\mu S_{2,k-1}} \ldots e^{-i\mu S_{2,1}}$$

$$\mathcal{J}_2^\pm = \sum_{k=1}^N e^{-i\mu S_{2,N}} \ldots e^{-i\mu S_{2,k+1}} J_{2,k}^\pm e^{-i\mu S_{3,k-1}} \ldots e^{-i\mu S_{3,1}} \tag{4.5}$$

$$\mathcal{J}_3^\pm = \sum_{k=1}^N (e^{-i\mu S_{1,N}} \ldots e^{-i\mu S_{1,k+1}} J_{3,k}^\pm + p\mathcal{J}_{1,k+1}^\pm J_{2,k}^\pm) e^{-i\mu S_{3,k-1}} \ldots e^{-i\mu S_{3,1}}.$$

Then from equation (2.15) for $K^-(\lambda) = 1$ and $K^+(\lambda) = M$, (2.20) and (4.4) we conclude that the leading asymptotic behaviour of the transfer matrix has the following form:

$$t(\lambda) \sim (\tfrac{1}{2})^{2N} e^{-2\mu\lambda N} (e^{-2i\mu-2i\mu\mathcal{S}_1} + e^{-2i\mu} p^2 \mathcal{J}_1^- \mathcal{J}_1^+ + e^{-2i\mu} p^2 \mathcal{J}_3^- \mathcal{J}_3^+$$

$$+ e^{2i\mu-2i\mu\mathcal{S}_3} + p^2 \mathcal{J}_2^- \mathcal{J}_2^+ + e^{-2i\mu\mathcal{S}_2}). \tag{4.6}$$

We introduce the operators M_1 and M_2:

$$\mathcal{S}_1 = N - M_1 \qquad \mathcal{S}_2 = M_1 - M_2 \qquad \mathcal{S}_3 = M_2. \tag{4.7}$$

We consider simultaneous eigenstates of M_i and the transfer matrix, i.e.

$$M_i|\Lambda^{(m)}\rangle = m_i|\Lambda^{(m)}\rangle \qquad t(\lambda)|\Lambda^{(m)}\rangle = \Lambda^{(m)}(\lambda)|\Lambda^{(m)}\rangle \tag{4.8}$$

(to simplify our notation we write (m) instead of $(m_1 m_2)$). We choose these states to be annihilated by $\mathcal{J}_i^+$:

$$\mathcal{J}_i^+|\Lambda^{(m)}\rangle = 0. \tag{4.9}$$

We conclude, from (4.6), (4.8) and (4.9), that the asymptotic behaviour of the corresponding eigenvalue is given by

$$\Lambda^{(m)}(\lambda) \sim (\tfrac{1}{2})^{2N} e^{-2\mu\lambda N} (e^{-2i\mu(1+N-m_1)} + e^{2i\mu(1-m_2)} + e^{-2i\mu(m_1-m_2)}). \tag{4.10}$$

In order to determine the asymptotic behaviour of $\bar{t}(\lambda)$, we also need the asymptotic behaviour of $R_{\bar{0}k}(\lambda)(R_{k\bar{0}}(\lambda))$ for $\lambda \to -\infty$:

$$R_{\bar{0}k} \sim \tfrac{1}{2} e^{-\mu\lambda-\frac{3i\mu}{2}} \begin{pmatrix} e^{i\mu S_{3,k}} & qp J_{2,k}^- & -q^2 p J_{3,k}^- \\ 0 & e^{i\mu S_{2,k}} & qp J_{1,k}^- \\ 0 & 0 & e^{i\mu S_{1,k}} \end{pmatrix}$$

$$R_{k\bar{0}} \sim \tfrac{1}{2} e^{-\mu\lambda-\frac{3i\mu}{2}} \begin{pmatrix} e^{i\mu S_{3,k}} & 0 & 0 \\ qp J_{2,k}^+ & e^{i\mu S_{2,k}} & 0 \\ -q^2 p J_{3,k}^+ & qp J_{1,k}^+ & e^{i\mu S_{1,k}}. \end{pmatrix} \tag{4.11}$$

We define $R_{\bar{0}k}(\lambda)$ from (2.6) using

$$V = \begin{pmatrix} & & q \\ & -1 & \\ q^{-1} & & \end{pmatrix} \tag{4.12}$$

where $q = e^{i\mu}$, and the asymptotic behaviour of $\bar{t}(\lambda)$ is given by

$$\bar{t}(\lambda) \sim (\tfrac{1}{2})^{2N} e^{-2\mu\lambda N - 3i\mu N} (e^{2i\mu + 2i\mu S_1} + e^{2i\mu} p^2 \bar{\mathcal{J}}_1^- \bar{\mathcal{J}}_1^+ + e^{2i\mu} p^2 \bar{\mathcal{J}}_3^- \bar{\mathcal{J}}_3^+$$

$$+ e^{-2i\mu + 2i\mu S_3} + p^2 \bar{\mathcal{J}}_2^- \bar{\mathcal{J}}_2^+ + e^{2i\mu S_2}) \tag{4.13}$$

with

$$\bar{\mathcal{J}}_1^\pm = \sum_{k=1}^{N} e^{i\mu S_{2,N}} \dots e^{i\mu S_{2,k+1}} J_{1,k}^\pm e^{i\mu S_{1,k-1}} \dots e^{i\mu S_{1,1}}$$

$$\bar{\mathcal{J}}_2^\pm = \sum_{k=1}^{N} e^{i\mu S_{3,N}} \dots e^{i\mu S_{3,k+1}} J_{2,k}^\pm e^{i\mu S_{2,k-1}} \dots e^{i\mu S_{2,1}} \tag{4.14}$$

$$\bar{\mathcal{J}}_3^\pm = \sum_{k=1}^{N} (e^{i\mu S_{3,N}} \dots e^{i\mu S_{3,k+1}} J_{3,k}^\pm - p \bar{\mathcal{J}}_{2,k+1}^\pm J_{1,k}^\pm) e^{i\mu S_{1,k-1}} \dots e^{i\mu S_{1,1}}.$$

The $|\Lambda^{(m)}\rangle$ states are also annihilated by $\bar{\mathcal{J}}_i^+$, i.e.

$$\bar{\mathcal{J}}_i^+ |\Lambda^{(m)}\rangle = 0 \tag{4.15}$$

where the corresponding eigenvalue of $\bar{t}(\lambda)$ is

$$\bar{\Lambda}^{(m)}(\lambda) \sim (\tfrac{1}{2})^{2N} e^{-2\lambda\mu N - 3i\mu N} (e^{2i\mu(1+N-m_1)} + e^{-2i\mu(1-m_2)} + e^{2i\mu(m_1-m_2)}). \tag{4.16}$$

We consider the state with all 'spins' up, i.e.

$$|\Lambda^{(0)}\rangle = \bigotimes_{k=1}^{N} |+\rangle_{(k)}. \tag{4.17}$$

This is annihilated by $\mathcal{J}_i^+$ and $\bar{\mathcal{J}}_i^+$, where (we suppress the (k) index):

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.18}$$

We assume this is an eigenstate of the transfer matrix and it is also an eigenstate of $\bar{t}(\lambda)$. The action of the R-matrix on the $|+\rangle$ state gives lower and upper triangular matrices, i.e.

$$\langle +| R_{0k}(\lambda) = \langle +| \begin{pmatrix} A_k & 0 & 0 \\ C_{1,k} & D_{1,k} & 0 \\ C_{2,k} & D_{3,k} & D_{4,k} \end{pmatrix}$$

$$R_{k0}(\lambda)|+\rangle = \begin{pmatrix} A_k & B_{1,k} & B_{2,k} \\ 0 & D_{1,k} & D_{2,k} \\ 0 & 0 & D_{4,k} \end{pmatrix} |+\rangle \tag{4.19}$$

where the matrices A, B_i, C_i, D_i act on the quantum space and they are determined by the form of the R-matrix (2.19). Then the action of the transfer matrix on the pseudo-vacuum is

$$\langle \Lambda^{(0)}| T_0(\lambda) = \langle \Lambda^{(0)}| \begin{pmatrix} \mathcal{A} & 0 & 0 \\ \mathcal{C}_1 & \mathcal{D}_1 & 0 \\ \mathcal{C}_2 & \mathcal{D}_3 & \mathcal{D}_4 \end{pmatrix} \qquad \hat{T}_0(\lambda)|\Lambda^{(0)}\rangle = \begin{pmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ 0 & \mathcal{D}_1 & \mathcal{D}_2 \\ 0 & 0 & \mathcal{D}_4 \end{pmatrix} |\Lambda^{(0)}\rangle \tag{4.20}$$

and the matrix elements are given by

$$\mathcal{A} = A_N \dots A_1 \qquad \mathcal{D}_1 = D_{1,N} \dots D_{1,1} \qquad \mathcal{D}_4 = D_{4,N} \dots D_{4,1} \tag{4.21}$$

$$\mathcal{C}_1 = \sum_{k=1}^{N} D_{1,N} \dots D_{1,k+1} C_{1,k} A_{k-1} \dots A_1 \qquad \mathcal{C}_0 = 0$$

$$\mathcal{D}_3 = \sum_{k=1}^{N} D_{4,N} \dots D_{4,k+1} D_{3,k} D_{1,k-1} \dots D_{1,1} \qquad \mathcal{D}_{3,0} = 0 \tag{4.22}$$

$$\mathcal{C}_2 = \sum_{k=1}^{N} D_{4,N} \dots D_{4,k+1} (C_{2,k} A_{k-1} \dots A_1 + D_{3,k} \mathcal{C}_{1,k-1}).$$

Similarly, to find $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{D}_2$ we replace $\mathcal{C}_1$, $\mathcal{D}_3$, $\mathcal{C}_2$ and $\mathcal{C}_1$ with $\mathcal{B}_1$, $\mathcal{D}_2$, $\mathcal{B}_2$ and $\mathcal{B}_1$ correspondingly. The transfer matrix eigenvalue for the pseudo-vacuum state is

$$\Lambda^{(0)}(\lambda) = \langle \Lambda^{(0)} | (\mathrm{e}^{-2\mathrm{i}\mu} \mathcal{A}^2 + \mathcal{D}_1^2 + \mathrm{e}^{2\mathrm{i}\mu} \mathcal{D}_4^2 + \mathcal{C}_1 \mathcal{B}_1 + \mathrm{e}^{2\mathrm{i}\mu} \mathcal{C}_2 \mathcal{B}_2 + \mathrm{e}^{2\mathrm{i}\mu} \mathcal{D}_3 \mathcal{D}_2) | \Lambda^{(0)} \rangle \tag{4.23}$$

and after some tedious algebra we find

$$\Lambda^{(0)}(\lambda) = f(\lambda) \left(a(\lambda)^{2N} \frac{\sinh 2\mu(\lambda + \mathrm{i})}{\sinh(2\mu\lambda)} + b(\lambda)^{2N} \left(1 + \frac{\sinh \mu(2\lambda + \mathrm{i})}{\sinh \mu(2\lambda + 3\mathrm{i})} \right) \right). \tag{4.24}$$

We make the assumption that a general eigenvalue has the form of a 'dressed' pseudo-vacuum eigenvalue, i.e.

$$\Lambda^{(m)}(\lambda) = f(\lambda) \left(a(\lambda)^{2N} \frac{\sinh 2\mu(\lambda + \mathrm{i})}{\sinh(2\mu\lambda)} A_1(\lambda) + b(\lambda)^{2N} \left(A_2(\lambda) + \frac{\sinh \mu(2\lambda + \mathrm{i})}{\sinh \mu(2\lambda + 3\mathrm{i})} A_3(\lambda) \right) \right). \tag{4.25}$$

Again, the action of $R_{\bar{0}k}(\lambda)$ on the $|+\rangle$ state gives upper and lower triangular matrices (see (4.19)), so we find an analogous equation for $\bar{\Lambda}^{(m)}(\lambda)$,

$$\bar{\Lambda}^{(m)}(\lambda) = f(\lambda) \left(\bar{a}(\lambda)^{2N} \frac{\sinh \mu(2\lambda + \mathrm{i})}{\sinh \mu(2\lambda + 3\mathrm{i})} \bar{A}_1(\lambda) + \bar{b}(\lambda)^{2N} \left(\bar{A}_2(\lambda) + \frac{\sinh 2\mu(\lambda + \mathrm{i})}{\sinh(2\mu\lambda)} \bar{A}_3(\lambda) \right) \right) \tag{4.26}$$

(we suppress the (m) index from the 'dressing' functions), where

$$f(\lambda) = \frac{\sinh(2\mu\lambda) \sinh \mu(2\lambda + 3\mathrm{i})}{\sinh 2\mu(\lambda + \mathrm{i}) \sinh \mu(2\lambda + \mathrm{i})} \tag{4.27}$$

$$a(\lambda) = \sinh \mu(\lambda + \mathrm{i}) \qquad b(\lambda) = \sinh(\mu\lambda) \tag{4.28}$$

and $\bar{a}(\lambda)$, $\bar{b}(\lambda)$ are $a(-\lambda - \rho)$, $b(-\lambda - \rho)$, respectively. It is obvious that $\Lambda^{(0)}(\lambda) = \bar{\Lambda}^{(0)}(-\lambda - \rho)$. From the asymptotic behaviour of the transfer matrix we conclude

$$A_1(\lambda) \to \mathrm{e}^{2\mathrm{i}\mu m_1} \qquad A_2(\lambda) \to \mathrm{e}^{2\mathrm{i}\mu(m_2 - m_1)} \qquad A_3(\lambda) \to \mathrm{e}^{-2\mathrm{i}\mu m_2} \tag{4.29}$$

and

$$\bar{A}_1(\lambda) \to \mathrm{e}^{-2\mathrm{i}\mu m_1} \qquad \bar{A}_2(\lambda) \to \mathrm{e}^{-2\mathrm{i}\mu(m_2 - m_1)} \qquad \bar{A}_3(\lambda) \to \mathrm{e}^{2\mathrm{i}\mu m_2}. \tag{4.30}$$

We substitute the eigenvalues into the fusion equations (3.9) and we obtain conditions involving A_1, A_3 and $\bar{A}_1$, $\bar{A}_3$. It is clear that $\Lambda^{(0)}(\lambda)$ satisfies (3.9), a fact that further supports our assumption that $|\Lambda^{(0)}\rangle$ is an eigenstate of the transfer matrices, $t(\lambda)$ and $\bar{t}(\lambda)$. From the fusion equations we find that

$$A_1(\lambda + \rho) \bar{A}_1(\lambda) = 1 \qquad \bar{A}_3(\lambda + \rho) A_3(\lambda) = 1 \tag{4.31}$$

where notice that we obtain two equations from (3.9), one from $\bar{1}2$ fusion and one from $1\bar{2}$, whereas e.g. for the $A_1^{(1)}$ case we obtain only one such equation. From the crossing property of the transfer matrix (2.24),

$$A_i(-\lambda - \rho) = \bar{A}_i(\lambda) \qquad i = 1, 2, 3. \tag{4.32}$$

Combining the last two conditions we find

$$A_1(\lambda) A_1(-\lambda) = 1 \qquad \bar{A}_3(\lambda) \bar{A}_3(-\lambda) = 1. \tag{4.33}$$

Note that the previous equations mix the 'dressing' functions of $\Lambda^{(m)}(\lambda)$ and $\bar{\Lambda}^{(m)}(\lambda)$, which is expected because of the form of equations (2.24) and (3.9). In the case of a model with crossing symmetry, e.g. $A_1^{(1)}$, the two eigenvalues become degenerate. From the periodicity of the transfer matrix

$$t\left(\lambda + \frac{\mathrm{i}\pi}{\mu} \right) = t(\lambda) \tag{4.34}$$

we expect the eigenvalues to be periodic as well. We impose that $A_2(\lambda)$ $(\bar{A}_2(\lambda))$ have the same poles as $A_1(\lambda)$ and $A_3(\lambda)$ $(\bar{A}_1(\lambda)$ and $\bar{A}_3(\lambda))$. Also. the residue of $\Lambda^{(m)}(\Lambda)$ at $\lambda = -i$ should vanish; thus we obtain the condition

$$A_2(-i) = A_3(-i).$$

We put all the above requirements together and we find that

$$A_1(\lambda) = \prod_{j=1}^{m_1} \frac{\sinh \mu(\lambda + \lambda_j^{(1)})}{\sinh \mu(\lambda + \lambda_j^{(1)} + i)} \frac{\sinh \mu(\lambda - \lambda_j^{(1)} - i)}{\sinh \mu(\lambda - \lambda_j^{(1)})} \tag{4.35}$$

$$A_2(\lambda) = \prod_{j=1}^{m_1} \frac{\sinh \mu(\lambda + \lambda_j^{(1)} + 2i)}{\sinh \mu(\lambda + \lambda_j^{(1)} + i)} \frac{\sinh \mu(\lambda - \lambda_j^{(1)} + i)}{\sinh \mu(\lambda - \lambda_j^{(1)})}$$
$$\times \prod_{j=1}^{m_2} \frac{\sinh \mu(\lambda + \lambda_j^{(2)} + i)}{\sinh \mu(\lambda + \lambda_j^{(2)} + 2i)} \frac{\sinh \mu(\lambda - \lambda_j^{(2)} - i)}{\sinh \mu(\lambda - \lambda_j^{(2)})} \tag{4.36}$$

$$A_3(\lambda) = \prod_{j=1}^{m_2} \frac{\sinh \mu(\lambda + \lambda_j^{(2)} + 3i)}{\sinh \mu(\lambda + \lambda_j^{(2)} + 2i)} \frac{\sinh \mu(\lambda - \lambda_j^{(2)} + i)}{\sinh \mu(\lambda - \lambda_j^{(2)})}. \tag{4.37}$$

We obtain $\bar{A}_i(\lambda)$ from (4.32). It is easy to check that the eigenvalues satisfy all the above conditions. Moreover, we want the eigenvalues to be analytical, so the poles must vanish. This condition leads to the Bethe ansatz equations

$$e_1(\lambda_i^{(1)})^{2N} = \prod_{i \neq j=1}^{m_1} e_2(\lambda_i^{(1)} - \lambda_j^{(1)}) e_2(\lambda_i^{(1)} + \lambda_j^{(1)}) \prod_{j=1}^{m_2} e_{-1}(\lambda_i^{(1)} - \lambda_j^{(2)}) e_{-1}(\lambda_i^{(1)} + \lambda_j^{(2)})$$
$$1 = \prod_{i \neq j=1}^{m_2} e_2(\lambda_i^{(2)} - \lambda_j^{(2)}) e_2(\lambda_i^{(2)} + \lambda_j^{(2)}) \prod_{j=1}^{m_1} e_{-1}(\lambda_i^{(2)} - \lambda_j^{(1)}) e_{-1}(\lambda_i^{(2)} + \lambda_j^{(1)}) \tag{4.38}$$

where we have defined $e_n(\lambda)$ as

$$e_n(\lambda) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})}. \tag{4.39}$$

The exact computation for the general case becomes complicated; however, one can 'guess' the form of the general eigenvalue, having in mind all the conditions that it must satisfy. The expression for the spectrum of the transfer matrix for any $\mathcal{N}$ is given by

$$\Lambda^{(m)}(\lambda) \propto b(\lambda)^{2N} \sum_{k=1}^{\mathcal{N}} \frac{\sinh(2\mu\lambda)}{\sinh \mu(2\lambda + (k-1)i)} \frac{\sinh \mu(2\lambda + i)}{\sinh \mu(2\lambda + ki)} A_k(\lambda) \tag{4.40}$$

where

$$A_k(\lambda) = \prod_{j=1}^{m_{k-1}} \frac{\sinh \mu(\lambda + \lambda_j^{(k-1)} + ki)}{\sinh \mu(\lambda + \lambda_j^{(k-1)} + (k-1)i)} \frac{\sinh \mu(\lambda - \lambda_j^{(k-1)} + i)}{\sinh \mu(\lambda - \lambda_j^{(k-1)})}$$
$$\times \prod_{j=1}^{m_k} \frac{\sinh \mu(\lambda + \lambda_j^{(k)} + (k-1)i)}{\sinh \mu(\lambda + \lambda_j^{(k)} + ki)} \frac{\sinh \mu(\lambda - \lambda_j^{(k)} - i)}{\sinh \mu(\lambda - \lambda_j^{(k)})}$$
$$k = 1, \ldots, \mathcal{N} \tag{4.41}$$

$m_0 = N$ and $m_{\mathcal{N}} = 0$. Also, $\bar{\Lambda}^{(m)}(\lambda) = \Lambda^{(m)}(-\lambda - \rho)$. The procedure we described uniquely fixes the 'dressing' functions. By inspection we can verify that the above eigenvalues indeed

satisfy all the requirements we derived previously, i.e. analyticity, asymptotic behaviour, crossing etc, e.g.

$$\bar{A}_k(\lambda) = A_k(-\lambda - \rho) \qquad k = 1, \ldots, \mathcal{N} \qquad (4.42)$$

$$A_1(\lambda)A_1(-\lambda) = 1 \qquad \bar{A}_{\mathcal{N}}(\lambda)\bar{A}_{\mathcal{N}}(-\lambda) = 1. \qquad (4.43)$$

We can see from (4.41) that every two terms have the same poles. From the analyticity of the eigenvalues we obtain the Bethe ansatz equations:

$$1 = \prod_{i \neq j = 1}^{m_k} e_2(\lambda_i^{(k)} - \lambda_j^{(k)})e_2(\lambda_i^{(k)} + \lambda_j^{(k)}) \prod_{j=1}^{m_{k+1}} e_{-1}(\lambda_i^{(k)} - \lambda_j^{(k+1)})e_{-1}(\lambda_i^{(k)} + \lambda_j^{(k+1)})$$

$$\times \prod_{j=1}^{m_{k-1}} e_{-1}(\lambda_i^{(k)} - \lambda_j^{(k-1)})e_{-1}(\lambda_i^{(k)} + \lambda_j^{(k-1)}) \qquad k = 1, \ldots, \mathcal{N}. \qquad (4.44)$$

For $\mathcal{N} = 3$ we recover (4.38). The results, as expected, coincide with the known ones obtained by nesting [3, 4] (for $\xi \to \pm i\infty$).

5. Discussion

We have generalized the fusion procedure for open spin chains without crossing symmetry. Furthermore, we have shown that even though the R-matrix does not have crossing symmetry the transfer matrix satisfies a crossing property (2.24). We have applied these results to diagonalize the transfer matrix via the analytical Bethe ansatz method. We have found explicit expressions for the transfer matrix spectrum (4.40) and we have deduced the Bethe ansatz equations (4.44), avoiding nesting. The main realization in this paper was the necessity of the transfer matrix $\bar{t}(\lambda)$ (2.25) in the derivation of the analytical Bethe ansatz. Indeed, it was necessary to consider $\bar{t}(\lambda)$ together with the usual transfer matrix in order to derive the conditions that the eigenvalues should satisfy.

Here, we have considered the special case where the chain has a $U_q(SU(\mathcal{N}))$ symmetry. However, we believe that the previous analysis can be extended even in the case of the reduced symmetry $U_q(SU(l)) \times U_q(SU(\mathcal{N} - l)) \times U(1)$ [26]. Moreover, the Bethe ansatz equations are known for open spin chains with 'soliton preserving' boundary conditions. There is also the case of 'soliton non-preserving' boundary conditions (see, e.g., [22, 27]) for which the Bethe ansatz equations are not known. Using the analytical Bethe ansatz, one can presumably derive the corresponding transfer matrix spectrum and the Bethe ansatz equations, avoiding nesting. We hope to address these questions in a future work [28].

Acknowledgments

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